Historical Perspectives on the Purposes of School Algebra

Nerida F. Ellerton
Illinois State University
<ellerton@ilstu.edu >

Sinan Kanbir
University of Wisconsin
<Sinan.Kanbir@uwsp.edu>

M. A. (Ken) Clements
Illinois State University
<clements@ilstu.edu>

In this paper, we identify, from historical vantage points, the following six purposes of school algebra: (a) algebra as a body of knowledge essential to higher mathematical and scientific studies, (b) algebra as generalised arithmetic, (c) algebra as a prerequisite for entry to higher studies, (d) algebra as offering a language and set of procedures for modelling real-life problems, (e) algebra as an aid to describing structural properties in elementary mathematics, and (f) algebra as a study of variables. We conclude with brief commentary on the question whether school algebra is a unidimensional field of study.

Introduction

Published summaries of aspects of the history of school algebra in the eighteenth and nineteenth centuries have mainly comprised analyses of school algebra textbooks written by European or American authors (see, e.g., Artigue, Assude, Grugeon, & Lenfant, 2001; Chateauneuf, 1929; Chazan, 2012; Chorlay, 2011; da Ponte & Guimarães, 2014; Schubring, 2011). No comprehensive international history of school algebra has ever been published. In this paper, we outline a framework which identifies six purposes for school algebra which have emerged over the past 350 years. We also comment on some of the theories which have been proposed for explaining why school algebra has caused so much difficulty for so many learners.

Algebra in Secondary School Mathematics—In the Beginning

Algebra made a late entry into the canonical school mathematics curriculum, having long been preceded by arithmetic and Euclidean geometry (da Ponte & Guimarães, 2014). Ellerton and Clements (2017) have argued that secondary school mathematics first appeared as a definite curriculum component in 1673, at Christ's Hospital, a school for poor children located in central London. In 1693, algebra formally became a component of the mathematics curriculum at Christ's Hospital when that school published a book on algebra which had been written in Latin by a Swiss mathematician, Johannis Alexandri (1693). That book became the algebra reference text for 14- to 16-year-old boys attending the Royal Mathematical School (hereafter "RMS") within Christ's Hospital program. RMS was established in 1673 for boys who would be selected, from among the academically most capable boys in the School, to prepare them for careers in the Royal Navy or in the merchant marine.

In 1694, Isaac Newton recommended to Christ's Hospital that the RMS mathematics curriculum should include "artificial arithmetic"—another name for "algebra" (Turnbull, 1961). Although the school adopted this recommendation, it insisted that the RMS boys should complete 4½ years of formal study in Latin *before* beginning to study "artificial arithmetic". The RMS boys would be required to leave school at the age of 16, and would only be allowed to study mathematics for 1½ years. If one looks carefully at Alexandri's (1693) book—used at Christ's Hospital between 1693 and 1715—it becomes clear that the

(2017). In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 221-228). Melbourne: MERGA.

algebra which the boys were asked to study was pitched at a very high level (Ellerton & Clements, 2017). That was true, despite the fact that most of the boys had had no previous instruction in algebra before moving into the RMS program.

The requirement by Christ's Hospital authorities that RMS boys would be allowed to spend 1½ years only studying mathematics, and that that would come after they had spent 4½ years studying Latin, might seem to be bizarre to modern educators. However, the "best" educational thinking of the time insisted that any worthwhile form of post-elementary education should emphasise Latin and Greek. Proceeding from that assumption, the Secretary to the Admiralty, Samuel Pepys (of diary fame), was able to persuade Christ's Hospital that even during the 18 months when the RMS boys would study mathematics they should do nightly exercises in Latin under the supervision of the school's classics masters.

Future implications for school algebra arising from Christ's Hospital's RMS model should not go unnoticed. By adopting Pepys's and Newton's plan, Christ's Hospital made a policy decision that only very capable boys should study mathematics beyond arithmetic, and that the best curricular preparation for studying school mathematics was a solid 4½-year block of Latin, when the boys were aged between 9½ and 14 years. During the 18 months when the RMS boys would be allowed to study mathematics they would be expected to learn, among other things, algebra, Euclidean geometry, plane trigonometry, and spherical trigonometry.

That well-educated people would agree to such an intended curriculum not only testified to the high level of faith in the virtues of a classical education but also to a belief that algebra, geometry and trigonometry would be relatively easy for students who had already mastered the classics. It was also assumed that the study of school algebra should be confined to very capable boys. A similar attitude would prevail in education circles across Europe and in North America throughout the eighteenth century and for much of the nineteenth century, and would be translated to the colonies. In the second half of the nineteenth century in all the early Australian universities—in Sydney, Melbourne, Tasmania, and Adelaide—matriculation passes in Latin, Greek, Arithmetic, Geometry and Algebra were required before a student could graduate, including those who were destined to become clergymen and lawyers (Clements, 1979; French, 1956).

During the 18th century, largely as a result of the example of Christ's Hospital and of the influence of writings on algebra by distinguished European mathematicians like Étienne Bézout (1794), Alexis Clairaut (1746) and Leonhard Euler (1770), algebra filtered into the curricula of many secondary schools (which were often called "academies") in Europe and America (da Ponte & Guimarāes, 2014). However, in almost all of the algebra textbooks which were written by highly-regarded mathematicians, algebra was presented in the form of generalised arithmetic. There were exceptions—at Christ's Hospital, for example, James Hodgson (1723) maintained that school algebra was predominantly something which should help school students solve practical problems. Hodgson had been appointed master of RMS in 1709, and he immediately protested against the practice, in the school, of teaching algebra in Latin. Despite the opposition of Isaac Newton, Hodgson convinced Christ's Hospital authorities that RMS students did not know Latin well enough to be able to learn algebra in that language. From 1710, RMS students learned algebra in English.

Although Hodgson recognised that a stipulation that 14-to-16-year-olds should learn algebra from a text written in Latin was foolish, that did not stop him from demanding algebra at a very high level from his students. Early in his *A System of Mathematics,* Hodgson (1723) freely used Newton's version of differential calculus ("fluxions"), and throughout the book he proved theorems using algebraic methods. Undoubtedly, Hodgson's high standards were related to the fact that RMS boys could not graduate unless they passed an intensive verbal examination of their mathematical knowledge conducted by disinterested experts—and it could be argued that those expectations generated unrealistically high standards for school algebra students, both at the time of Hodgson and in the future.

School Algebra in the Eighteenth and Nineteenth Centuries

During the Hodgson era at Christ's Hospital (1709–1755), the Royal Mathematical School came to be recognised within Europe as a leader in school mathematics (Ellerton & Clements, 2017; Hans, 1951). This persuaded other schools, especially those which prepared students to become surveyors, or officers in the navy, merchant marine, or the army, to introduce algebra and trigonometry into their curricula. A top-down model of curriculum development manifested itself, and most of the algebra textbooks used in schools were authored by professional mathematicians (e.g., Bézout, 1794; Bonnycastle, 1788; Clairaut, 1746; Todhunter, 1863; Wolfe, 1739).

The 18th and 19th centuries were marked by massive colonisation programs by leading European nations, and it was not surprising that the colonisers tended to introduce school mathematics textbooks into their colonies which had been written in the "home" nations. Not only were the languages used in those textbooks those of the colonising powers, but also the authors were, almost always, based in Europe. Inevitably, the textbooks were written in a way which suggested that school mathematics should be a culture-free exercise. Even for students in the European homelands, most algebra textbooks were designed to suit the perceived needs of children of elites. The first algebra textbooks used in the "colonies", around the world, were often written from high mathematics vantage points and were unsuited to the needs of indigenous children, of children of slaves, or of children whose command of the forms of language used by mathematics teachers and authors of mathematics textbooks was not strong (Clements, Grimison, & Ellerton, 1989).

Table 1, adapted from Kanbir, Clements, and Ellerton (in press), outlines six "purposes" of school algebra which emerged from analyses of data sets in numerous archives, including handwritten and printed documents. Table 1 is framed historically by the dates given in the first column. The first two named purposes are associated with periods which began late in the 17th century or early in the 18th century. Then, late in the 18th century and early in the 19th century, school algebra began to be linked to entrance requirements of higher-education institutions—even for courses such as divinity, and law. From the second half of the 19th century, there was recognition of a need to introduce a kind of school algebra which could model, and therefore help to solve, real-life problems.

At about the same time as the idea emerged that algebra should facilitate mathematical modelling of real-life situations, there arose a movement by which school algebra should be important in enabling students to recognise the structure of the real-number system. Finally, in the second half of the twentieth century, especially at the time of the "new mathematics", the importance of the concept of a variable, and of the power of that concept to summarise major mathematical ideas and to model real-life situations, began to be emphasised.

Table 1
Six Purposes for Secondary School Algebra (Adapted from Kanbir et al., in press)

Period	Assumed Purpose	Key Writers or Players	Comments
1693–2017	*Purpose 1*: Knowledge essential for higher mathematics and science.	R. Descartes, H. Ditton, I. Newton, G. Leibniz, S. Lacroix	Mathematicians noted that algebra could assist students to comprehend and express concepts and principles in higher mathematics (e.g., in conic sections, trigonometry, calculus, and mechanics).
1700–2017	*Purpose 2*: Generalised arithmetic	L. Bourdon, E. Bézout, L. Euler, N. Pike, I. Todhunter	Emphasis was on the syntax and the semantics of elementary algebra. Solving an equation was equivalent to "finding the unknown number(s)". Sometimes, an axiomatic approach to algebra was adopted.
1800–2017	*Purpose 3*: A pre-requisite for entry to higher studies	Persons defining university-entrance pre-requisite subjects	During the 19th century many countries required school students who intended, subsequently, to enter prestigious universities to succeed in a school subject designated "Algebra." This tended to change after the 1960s, although it is still the case in the USA.
1870–2017	*Purpose 4*: A language for modelling real-life problems	J. Hodgson, S. Lacroix, J. Perry, F. Klein, E. H. Moore	This has often been called the "functional-thinking approach to algebra". Students describe sequences recursively and explicitly, prepare tables of values, and plot and interpret Cartesian graphs.
1870–2017	*Purpose 5*: An aid for describing basic structural properties	F. Klein, N. Bourbaki, C. Gattegno, Z. P. Dienes	Klein applied function concepts, structural ideas, and associated symbolisms to geometry. In the 1950s and 1960s this was incorporated into school algebra. Gattegno and Dienes argued that elementary-school students could learn algebra before arithmetic, and that structural properties should be emphasised.
1960–2017	*Purpose 6*: A study of variables	SMSG authors, R. Davis, D. Chazan	Solving an equation (or inequality) was seen as finding values of a variable which would make an open sentence true, and which would make it false. Tables of values and Cartesian graphs were to be regarded as depicting relationships between variables. Structural properties (e.g., the distributive property), stated in algebraic language, were to be seen as statements involving variables.

A Theoretical Lens for Interpreting the History of School Algebra

Acceptance, both consciously and subconsciously, of the purposes identified in Table 1 can be linked to corresponding changes in the thinking of pure and applied mathematicians. Figure 1 suggests how school mathematics came to be linked with the developments in mathematics made at various times by mathematicians. Figure 1 is to be interpreted in terms of a "lag time" theoretical lens: after pure and applied mathematicians developed new areas of mathematics, some of those areas were ultimately introduced into school mathematics if and whenever that was appropriate.

Thus, for example, in the first quarter of the 17th century, John Napier and Henry Briggs developed the concept of a logarithm, and this was quickly introduced to, and applied powerfully by, navigators and surveyors. Then, from 1673 onwards, RMS school students were taught how to use logarithms to solve problems associated with navigation. From about 1720 onwards, Isaac Newton's fluxions—which Newton had first developed in the 1670s—began to be introduced to RMS students (Hodgson, 1723). Felix Klein, in his Erlangen program of the early 1870s, showed how algebraic structures and functions could be linked and, ultimately, in the 1960s, this idea came to be embodied in school curricula of the new mathematics period. Matrices, which featured non-commutative multiplication, would also find their way into school algebra programs in the 1960s.

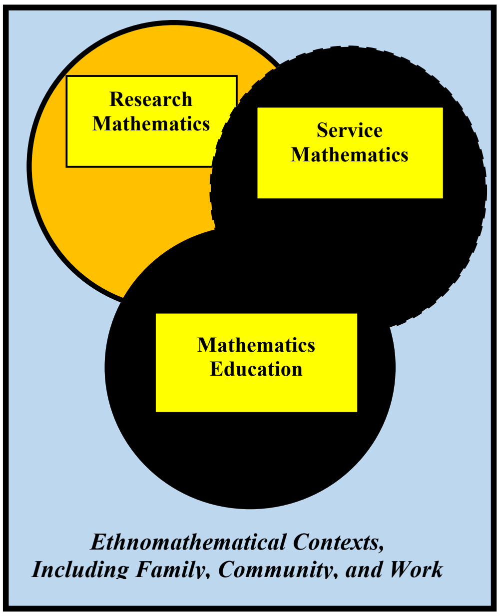

Figure 1. Different ways of "seeing" problems or situations that might relate to mathematics (Clements & Ellerton, 2015, p. 16).

Many aspects of school algebra are not a matter of great concern to mathematicians, and likewise, many parts of the higher forms of pure and applied mathematics are not of much interest to those primarily concerned with school algebra. But, as Figure 1 implies, there are intersections and, in particular, there is an intersection of pure mathematics, service mathematics, and mathematics education. Figure 1 also draws attention to how the purposes of school algebra, and the developments summarised in Table 1, were merged within ethnomathematical contexts, including contexts surrounding families, work situations, and communities in different nations.

Is School Algebra a Unidimensional Field of Study?

If, indeed, the six purposes listed in Table 1 are separable—as we claim they are—then important curriculum questions arise. Are all six purposes important in planning modern school algebra curricula? If the answer is "No," then which should be regarded as appropriate for which students, in which schools, and when, and why? Is school algebra a unidimensional field of study and, if it is not, what meaning should we give to the term "school algebra"? Do current school intended and implemented algebra curricula take sufficient account of the six purposes?

An obvious, and challenging, question is why six purposes, and not five? Or seven? Perhaps there is just one overarching purpose (such as "providing students with a language which will help them to learn to generalise"), which would synthesise the six purposes. Although the purposes identified in Table 1 are important from a mathematics curriculum perspective, it could be the case that there is no obvious best way of synthesising them adequately. One wonders which forms of words, and which conceptual structures, would be needed so that a description would be adequate.

In the 1960s and 1970s, when the "new mathematics" movement was sweeping the world, each Australian state decided to unify its mathematical offerings by creating composite subjects—often called, simply, "Mathematics" (or descriptive titles such as "Mathematics I", "Mathematics II", "Basic Mathematics", "Calculus", "Mechanics", and "Advanced Mathematics") (Clements, 2003). This did not occur in all parts of the world, however—for example, in the United States of America the titles Algebra I, Algebra II, and Geometry remained commonplace in all states. Even today, Algebra I and Algebra II are offered as distinct subjects across the United States—yet, such is the power of labeling that many U.S. students who have completed courses in calculus have told us that they do not think that algebra is involved in calculus (Ellerton & Clements, 2011).

Even in the late 1980s, it was still the case that, in some nations, many students did not study algebra at all (Clements, Keitel, Bishop, Kilpatrick, & Leung, 2013). In the second half of the 1980s in the United States of America, for example, about 30 percent of secondary-school students never studied algebra while still at school, and others studied it for only a short period of time. Yet, in Australia, and in some other nations, algebra was studied by almost all students in Grades 7 and 8 and, often, right through Grades 7 through 12 (Clements, 2003; Usiskin, 1988).

Algebra as generalised arithmetic has been, is, and will continue to be, important. That said, it is possible that the first steps in secondary-school algebra should *not* involve a strong emphasis on operating with algebraic symbols and other representations of varying quantities. Given the great difficulty which many beginning algebra students have always had with the language and written symbols of algebra, one must ask whether it is wise to insist that all seventh- (or eighth-) grade students should be expected to learn to recognise and use the main signifiers which point toward more sophisticated algebraic knowledge and skills (Kanbir et al., in press).

The push toward algebra as functional thinking, as an essential preliminary to being able to model real-life situations, is one that is often emphasised by mathematicians, mathematics educators, and scientists. But, one wonders how often most adults, even those who are well versed in algebra, use algebra to model practical problems—when was the last time you solved a quadratic equation outside of the classroom? On the other hand, graphical representations of data are common in most forms of media, and are much more accessible given today's technology. These representations often summarise relationships between varying quantities, and carry an invitation to express relationships between the variables. The challenge is for secondary teachers to devise ways and means of getting students to the point where they can identify variables relating to situations which are real and of interest to them, to devise learning contexts in which they generate data, and to assist the students to move towards making generalisations about how the data can be related. More than a century ago John Perry, in London, and Eliakim Hastings Moore, in Chicago, strongly favored that approach, but their efforts were thwarted by those who favored a more conservative approach to school mathematics (Kanbir et al., in press).

Too often, the directions of school algebra have been defined by outstanding mathematicians, like Isaac Newton, Felix Klein, and Jean Dieudonné, who have had little experience in teaching "ordinary" school children aged between 10 and 17 years. The new math(s) period generated textbooks which emphasised structure, but the language and symbolisms used in textbooks were so sophisticated that only a small percentage of school students engaged fully with the mathematical texts provided. In recent years, those who constructed the common-core algebra sequence in the United States seemed to think that structural properties (such as the commutative, associative and distributive properties) would be well known by middle-school students, but our experience and research indicates that that is definitely *not* the case (Kanbir et al., in press). Nevertheless, the question remains: Are algebraic structures too difficult for middle- and high-school students? Tzanakis (1991) is one among many to have argued that that is indeed the case.

We are not convinced that school algebra, as it is now being defined in various parts of the world, represents a unidimensional field of study. Studying algebra from a structural point of view may be quite a different thing from studying it from a modelling perspective. Certainly, the two can be combined, but the issue is whether students, teachers, and researchers, who are engaged in efforts to bring together the different strands recognise that although the two approaches may both be labeled as "algebra", they may not be representatives of "the same thing".

Ever since the introduction of algebra into the secondary school curriculum there has always been, among many—perhaps most—middle-school and lower-secondary-school students, a disconnect between the signifier (signs and symbols) and the signified (the mathematical objects which are being considered). We believe that the main issue is a semiotic one—not only do students need to understand what the authors of textbooks write, and their teachers say, about algebra, but they also need to learn how to express themselves using appropriate algebraic language, and to make generalisations of their own.

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